Classical Limit of the Casimir Entropy for Scalar Massless Field

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Abstract

We study the Casimir effect at finite temperature for a massless scalar field in the parallel plates geometry in N spatial dimensions, under various combinations of Dirichlet and Neumann boundary conditions on the plates. We show that in all these cases the entropy, in the limit where energy equipartitioning applies, is a geometrical factor whose sign determines the sign of the Casimir force.

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arXiv:0708.1615v1 [quant-ph] 12 Aug 2007

INTRODUCTION

The vacuum expectation value (VEV) of the Hamiltonian of a free scalar field in a large volume V (so that the allowed Fourier modes tend to a continuum) in N spatial dimensions at temperature zero is given by the following expression (given, e.g., in [27])

$$
\begin{aligned}
E_0 = \langle 0| \, H_{free} \, |0\rangle = \\
\int d^N x \int \frac{d^N k}{(2\pi)^N 2\omega_k} \left[\frac{1}{2} \left(\omega_k^2 + k^2 + m^2 \right) \right] = V \int \frac{d^N k}{(2\pi)^N} \frac{1}{2} \omega_k \\
\text{where } \omega_k^2 = k^2 + m^2, \ k^2 = \sum_{i=1}^{N} k_i^2,
\end{aligned}
\tag{1}
$$

and we use natural units $\hbar = c = k_B = 1$. This is the zero point energy of harmonic oscillators integrated over all momentum modes and over all space. The sum clearly diverges, but one may define a regularized subtracted hamiltonian $H' = H_{free} - E_0$. This simple shift is equivalent to normal ordering $H' =: H_{free} :$. In this way $\langle 0| \, H' \, |0\rangle = 0$ for a free field. Similarly, once the field is constrained by imposing boundary conditions and thus described by a different hamiltonian H, its vacuum expectation value $\tilde{E}_0 = \langle \tilde{0}| \, H \, |\tilde{0}\rangle$ diverges ($|\tilde{0}\rangle$ is the ground state of the constrained hamiltonian).

It is of physical interest to compute the difference $\tilde{E}_0 - E_0$ of the two (divergent) vacuum energies. This was first considered by Casimir (for the electromagnetic field) [9] and is therefore called the Casimir energy (and in general the subject is called the Casimir effect). As written, this difference is not well defined and thus requires regularization. The properly regularized finite part of that series is, by definition, the Casimir energy of the constrained system. This is carefully considered in [6, p 24] and [21, p 100]. We perform the regularization via the damping function method [6]. We regularize the energies of the bounded and of the unbounded system using the same regularizing parameter λ. In Eq.(2) we schematically designate this by

$$
E_{Casimir} = \left(\tilde{E}_0 - E_0 \right)_{reg}.
\tag{2}
$$

At the end of the calculation we take the limit $\lambda = 0$.

The Casimir energy depends on the geometry of the constraints. For example, in Casimir's original work [9] he considered two infinite parallel plates with separation d. In this case, the Casimir energy, $E_{Casimir}$, (per unit area of the plates) is a function of the separation and it gives rise to a force on the plates, the Casimir force,

$$
F_{Casimir} = -\frac{\partial E_{Casimir}}{\partial d}
\tag{3}
$$

and this leads to a measurable effect, the Casimir effect.

Various methods may be used to calculate Casimir energies, (e.g., the Green's function method [6],[8],[15],[16],[19],[20],[21]; path integration method [11],[22]; dimensional regularization method [10],[26]; mode summation method [9],[18],[19],[14],[22] etc.). There are several reviews and books on the subject, e.g., [6],[18],[19],[20],[21].

In the present work we consider the Casimir effect for a free massless scalar field in N spatial dimensions due to the presence of two parallel hyperplanes at distance d, at finite temperature. Hence, instead of the vacuum expectation values in the definition of the Casimir effect at zero temperature above, we have to consider expectation values with respect to the thermal equilibrium density matrix of the system.

We shall consider two cases. In the first case the scalar field is constrained to vanish on the hyperplanes, imitating the presence of ideal conductors (Dirichlet-Dirichlet (DD) boundary conditions (b.c.)). In the second case the field is set to zero on one hyperplane while the normal derivative of the field is set to zero on the other, thus imitating the presence of a perfect conductor and a perfectly permeable material (Dirichlet-Neumann (DN) b.c.).

In this paper we mostly apply the mode summation method. In the zero temperature case we apply the Green's function method as well. Our conclusion is that the DD case gives rise to attractive forces between the boundaries while the DN case gives rise to repulsive forces, a result which holds for both cases for any dimension and any $d > 0$ and $T \geq 0$. The case of Neumann-Neumann (NN) b.c. gives rise to the same effect as the DD case, as we shall see in the Green's function section. A case of particular interest is to obtain the force per unit area (pressure) on the boundaries in the high temperature limit (namely for T such that $\frac{T}{T_c} \gg 1$ where $T_c = \frac{\hbar c \pi}{k_B d}$).

A useful quantity, the Casimir entropy, may be defined [3],[7],[11],[24],[25]. Its sign in the high temperature limit determines whether the force on the boundary is attractive or repulsive (or zero if it vanishes). We obtain the high temperature limit of the Casimir entropy with the following results: For $N > 1$, in the DD case the Casimir entropy is negative (attractive force); in the DN case it is positive (repulsive force). For $N = 1$ we found that the Casimir entropy is zero in this high temperature limit for both DD and DN b.c. (the force between the boundary points tends to zero). Calculations for the Casimir energy for the DN case in 3 spatial dimensions were performed in [26] via zeta function regularization method. The Casimir energy in N spatial dimensions for the DD case was obtained in [1], [19]. We perform such calculation for the DN case in N spatial dimensions via the mode summation method and discuss the Casimir entropy in N spatial dimensions for both the DD and DN cases.

MODE SUMMATION AT ZERO TEMPERATURE

The dependence of the sign of the Casimir force on the boundary conditions is not yet fully understood. It was noted ([11], [24]) that for simple geometries in the classical limit (i.e., at temperatures wherein energy equipartitioning is applicable) the total Casimir force is entropic and the Casimir entropy depends solely on the geometry and the type of boundary conditions (by the latter we mean various combinations of Dirichlet and Neumann boundary conditions in the geometry of two parallel plates). We are interested in calculating the Casimir energy and free energy of a massless scalar field in the parallel plate geometry of two hyperplanes located at $x_N = 0$ and $x_N = d$. Consider N dimensional space with coordinate vector $\vec{x} = (x_1, ..., x_N)$ and a cube of edge length L in this space with faces given by $x_i = -\frac{L}{2}, \frac{L}{2}$ ($i = 1, ..., N - 1$) and $x_N = -(\frac{L-d}{2}), (\frac{L+d}{2})$ chosen in order to make our volume of interest a finite one. Later we will take $L \to \infty$, so that only the boundary conditions (b.c.) on $x_N = 0, d$ will be important (the reason of treating the coordinate x_N slightly differently is a matter of convenience).

The hyperplanes at $x_N = 0$ and $x_N = d$ with b.c. on them divide the cube into three regions $-(\frac{L-d}{2}) < x_N < 0, 0 < x_N < d, d < x_N < (\frac{L+d}{2})$. We say that our system is subjected to DD b.c. if on the hyperplanes $x_N = 0, d$ the scalar field is constrained by the conditions $\phi(t, \vec{x})|_{x_N=0} = \phi(t, \vec{x})|_{x_N=d} = 0$. Similarly we say that our system is subjected to DN b.c. if the scalar field fulfils the equations $\phi(t, \vec{x})|_{x_N=0} = \partial_{x_N} \phi(t, \vec{x})|_{x_N=d} = 0$. For convenience

the boundary conditions on the faces $x_i = -\frac{L}{2}, \frac{L}{2}$ $(i = 1, ..., N-1)$ are taken to be periodic, which we may write as $\phi(t, \vec{x})|_{x_i = -\frac{L}{2}} = \phi(t, \vec{x})|_{x_i = \frac{L}{2}}$ $(i = 1, ..., N-1)$. As for the remaining faces of the cube $(x_N = -(\frac{L-d}{2}), \frac{L+d}{2})$ we impose Dirichlet b.c. $\phi(t, \vec{x})|_{x_N = -(\frac{L-d}{2})} = \phi(t, \vec{x})|_{x_N = \frac{L+d}{2}} = 0$ for simplicity. We expect the details of the boundary conditions on the faces of the large cube to be unimportant in the limit $L \to \infty$.

Following our discussion prior to Eq.(2) , we define the Casimir energy (Casimir free energy) of the scalar field as the regularized difference (in the sense previously discussed) of the total mean energy (free energy) of the scalar field for the constrained and unconstrained cases. Introducing the Fourier modes of the field, each mode at temperature T contributes a mean energy

$$E_{\vec{k}}(T) = \frac{1}{2}\hbar\omega_k \coth(\frac{1}{2}\beta\hbar\omega_k) = \frac{1}{2}\hbar\omega_k + \frac{\hbar\omega_k}{e^{\beta\hbar\omega_k}-1} \tag{4}$$

(where $\beta = \frac{1}{k_B T}$) and free energy

$$F_{\vec{k}}(T) = \frac{1}{2}\hbar\omega_k + \frac{1}{\beta}\log(1 - \exp(-\beta\hbar\omega_k)). \tag{5}$$

At zero temperature both expressions coincide and reduce to

$$E_{\vec{k}}(0) = F_k(0) = \frac{1}{2}\hbar\omega_k \tag{6}$$

The angular frequency satisfies $\omega_k = ck$. The solutions of the free massless Klein Gordon equation $\Box\phi = 0$ in the whole space, for DD and DN b.c. on the hyperplanes $x_N = 0$ and $x_N = d$, give rise to the following wave vectors in the region $0 < x_N < d$:

$$\left\{ \begin{array}{c} \text{DD case: } \vec{k} = \left(\frac{2\pi}{L}n_1, ..., \frac{2\pi}{L}n_{N-1}, \frac{\pi}{d}n_N\right) \\ \text{DN case: } \vec{k} = \left(\frac{2\pi}{L}n_1, ..., \frac{2\pi}{L}n_{N-1}, \frac{\pi}{d}\left(n_N - \frac{1}{2}\right)\right) \end{array} \right\} \tag{7}$$
$$n_j = \{0, \pm 1, \pm 2, ...\} \text{ for } j = 1, ..., N-1 \text{ and } n_N = \{1, 2, ...\}.$$

Similar expressions hold in the other regions. The total mean energy (free energy) of a given system is just the regularized sum over the energies (free energies) of the individual modes. The Casimir energy is defined by

$$E_c(d, T) = \left\{ \sum_{\vec{k}_1} E_{\vec{k}_1}(T) + \sum_{\vec{k}_2} E_{\vec{k}_2}(T) + \sum_{\vec{k}_3} E_{\vec{k}_3}(T) - \sum_{\vec{k}_0} E_{\vec{k}_0}(T) \right\}_{reg} \tag{8}$$

and the Casimir free energy is:

$$F_c(d, T) = \left\{ \sum_{\vec{k}_1} F_{\vec{k}_1}(T) + \sum_{\vec{k}_2} F_{\vec{k}_2}(T) + \sum_{\vec{k}_3} F_{\vec{k}_3}(T) - \sum_{\vec{k}_0} F_{\vec{k}_0}(T) \right\}_{reg} \tag{9}$$

where the vectors $\vec{k}_1, \vec{k}_2, \vec{k}_3$ correspond to each one of the three regions (the order of regions is from negative x_N to positive) of the constrained system, and $\vec{k}_0$ corresponds to the unconstrained system. The sums in Eqs. (8) and (9) are discrete. But because L is large we can replace by integrals each of the sums whose summation index is multiplied by $\frac{2\pi}{L}$

or $\frac{2\pi}{L-d}$. Those $N-1$ components of $\vec{k}$ become continuous variables of integration and the integrations are performed over the whole space. In the continuum limit the Casimir energy (Casimir free energy) in both DD and DN cases is the difference of the total energy (free energy) of the constrained system in the volume between the planes and the energy (free energy) of the free system in the volume between the planes. The damping function we use is the exponential and its explicit form is given in Eq.(10) below. The Casimir energy at zero temperature due to DD b.c. is given by

$$E_c^{(DD)}(d,0) = \frac{L^{N-1}}{2(2\pi)^{N-1}} \times$$
$$(\sum_{n=1}^{\infty} \int d^{N-1}k \sqrt{k^2 + (\frac{\pi n}{d})^2} \exp(-\lambda\sqrt{k^2 + (\frac{\pi n}{d})^2}) -$$
$$\left(\frac{d}{\pi}\right) \int_0^{\infty} dk_N \int d^{N-1}k \sqrt{k^2 + k_N^2} \exp(-\lambda\sqrt{k^2 + k_N^2})) \tag{10}$$

where $k^2 = k_1^2 + ... + k_{N-1}^2$. The damping factor λ in Eq.(10) serves to regularize the otherwise divergent integrals; only at the end of the calculation do we take λ to zero. For the electromagnetic field we can introduce a cutoff damping factor for purely physical reasons because a physical conductor becomes transparent at high enough frequencies, namely frequencies higher than the plasma frequency of a given material. For a scalar field we may regard it as a mathematical device, since no scalar massless fields were observed in nature so far. The Casimir energy due to DN b.c. at zero temperature is obtained from (10) by shifting the summation index in the integrand $n \rightarrow n - \frac{1}{2}$. During the calculation d-independent terms arise, but we omit them since such d-independent terms don't influence physical observables such as force and pressure These terms may be interpreted as self-energy of the plates, since they are proportional to their area L^{N-1} . We denote shifted Casimir energy and shifted Casimir free energy by the same letters as the unshifted one. The Casimir energy at zero temperature due to DD b.c. which we obtain is:

$$E_c^{(DD)}(d,0) = -\frac{L^{N-1}}{(4\pi)^{\frac{N+1}{2}}d^N} \Gamma\left(\frac{N+1}{2}\right) \zeta(N+1) \tag{11}$$

(which agrees with [1],[10]). On the other hand, for DN b.c. the Casimir energy is:

$$E_c^{(DN)}(d,0) = \frac{L^{N-1}}{(4\pi)^{\frac{N+1}{2}}d^N} \Gamma\left(\frac{N+1}{2}\right) (1 - \frac{1}{2^N})\zeta(N+1) \tag{12}$$

(which agrees with [26]). The Casimir pressure for any b.c. at zero temperature is given by

$$P_c(d,0) = -\frac{1}{L^{N-1}} \left(\frac{\partial E_c(d,0)}{\partial d}\right). \tag{13}$$

We observe that Eq.(11) implies attraction between the hyperplanes, while Eq.(12) implies repulsion.

MODE SUMMATION AT FINITE TEMPERATURE

Let us now turn to the temperature dependent case. We are interested to obtain the expressions for Casimir energy, Casimir free energy and Casimir entropy in the cases of DD

and DN b.c.. As in the zero temperature case we consider the differences of energy (free energy) of the constrained system and energy (free energy) of the free system with mean energy (free energy) per mode given respectively by Eq.(4) and Eq.(5). We may decompose the total Casimir energy and total Casimir free energy into a sum of two terms: the Casimir energy at zero temperature (which we already know) and a remaining temperature dependent term. In order to obtain expressions for $E_c^{(DD)}(d,T)$ and $F_c^{(DD)}(d,T)$ and $E_c^{(DN)}(d,T)$, $F_c^{(DN)}(d,T)$ we use the Poisson summation formula [12]

$$\sum_{n=-\infty}^{\infty} \exp(2\pi inx) = \sum_{n=-\infty}^{\infty} \delta(x-n) \tag{14}$$

and our slightly modified version which is obtained by $x \to x + \frac{1}{2}$

$$\sum_{n=-\infty}^{\infty} (-1)^n \exp(2\pi inx) = \sum_{n=-\infty}^{\infty} \delta\left(x - \left(n - \frac{1}{2}\right)\right). \tag{15}$$

We drop divergent but d independent terms which occur since they don't change the pressure on the boundaries. After we perform the integrations and set $\lambda = 0$ we replace an infinite series by another infinite series. This new series expresses the result by known functions (McDonald functions K_ν [13]) and leads to some simplifications of the expressions, such as cancellation of $E_c(d,0)$, and makes easier the calculation of the high temperature limit ($\beta \to 0$). The Casimir energy of a massless scalar field in $N > 1$ dimensions at finite temperature due to DD and DN b.c. that we obtain is:

$$E_c^{(DD)}(d,T) = -\frac{L^{N-1}\pi}{2^{\frac{N}{2}-4}d^{\frac{N}{2}-2}\beta^{\frac{N}{2}+2}} \sum_{m,n=1}^{\infty} \frac{n^{\frac{N}{2}+1}}{m^{\frac{N}{2}-1}} K_{\frac{N}{2}-1}\left(\frac{4\pi dmn}{\beta}\right) \tag{16}$$

$$E_c^{(DN)}(d,T) = -\frac{L^{N-1}\pi}{2^{\frac{N}{2}-4}d^{\frac{N}{2}-2}\beta^{\frac{N}{2}+2}} \sum_{m,n=1}^{\infty} (-1)^m \frac{n^{\frac{N}{2}+1}}{m^{\frac{N}{2}-1}} K_{\frac{N}{2}-1}\left(\frac{4\pi dmn}{\beta}\right). \tag{17}$$

(d-dependent divergent terms arise during the calculation of the Casimir energy due to DD and DN b.c. for $N = 1$ but they cancel each other at the end. An independent short calculation based on (8) for the $N = 1$ case confirms that (16),(17) are valid for $N = 1$, too.) The double series converges for all $\beta \geq 0$ and $d > 0$, and as $\beta \to 0$ the series converges to zero. The conclusion is that the Casimir energy at high temperature tends to zero in accordance with the following heuristic argument [11]. Each mode carries mean energy given by Eq.(4). At the high temperature limit, namely for T such that $\frac{T}{T_c} \gg 1$ where $T_c = \frac{\hbar c\pi}{k_B d}$ is the geometry dependent temperature scale, we obtain $E_k(\beta \to 0) = \frac{1}{\beta} + O(\beta)$, i.e., each mode carries the same average energy (which equals the temperature by the well-known equipartition property of classical physics). The Casimir energy is the difference of sums of mean energies of the constrained and unconstrained systems. Moving the hyperplanes $x_N = 0, d$ adiabatically apart will change the energy levels but at each step there is one to one correspondence between the energy levels of the systems. Hence zero Casimir energy at the high temperature limit merely reflects the fact that the number of states of the constrained and free systems are equal. Generally the Casimir energy due to equipartition in the high temperature limit is of the form

$$E_c(d, T \to \infty) = (N_{constrained} - N_{free}) \times T \tag{18}$$

where $N_{constrained}$ and N_{free} are given by the integrals of the density of the modes $\rho_{constrained}$ and ρ_{free}, and correspond to the total number of states of the constrained and free systems. In our infinite parallel planes geometry the mode densities are given in [11] and $N_{constrained}$ and N_{free} turn out to be both infinite but equal in the sense that the right side of Eq.(18) is zero.

The Casimir free energy of a massless scalar field in $N > 1$ spatial dimensions due to DD and DN b.c is:

$$F_c^{(DD)}(d,T) = -\frac{\Gamma\left(\frac{N}{2}\right)L^{N-1}}{\pi^{\frac{N}{2}}2^N \beta d^{N-1}}\zeta(N) -$$
$$\frac{L^{N-1}}{2^{\frac{N}{2}-2}d^{\frac{N}{2}-1}\beta^{\frac{N}{2}+1}}\sum_{m,n=1}^{\infty}\frac{n^{\frac{N}{2}}}{m^{\frac{N}{2}}}K_{\frac{N}{2}}\left(\frac{4\pi mnd}{\beta}\right) \tag{19}$$

$$F_c^{(DN)}(d,T) = \frac{\Gamma\left(\frac{N}{2}\right)L^{N-1}}{\pi^{\frac{N}{2}}2^N \beta d^{N-1}}\left(1 - \frac{1}{2^{N-1}}\right)\zeta(N) -$$
$$\frac{L^{N-1}}{2^{\frac{N}{2}-2}d^{\frac{N}{2}-1}\beta^{\frac{N}{2}+1}}\sum_{m,n=1}^{\infty}(-1)^m\frac{n^{\frac{N}{2}}}{m^{\frac{N}{2}}}K_{\frac{N}{2}}\left(\frac{4\pi mnd}{\beta}\right) \tag{20}$$

(the last two formulas give the correct result for the $N = 1$ case, too (see also below); $\zeta(1)$ which is divergent is cancelled by an appropriate term in the double series). The Casimir pressure on the boundary for any b.c. is given by

$$P_c(d,T) = -\frac{1}{L^{N-1}}\left(\frac{\partial F_c(d,T)}{\partial d}\right)_T. \tag{21}$$

In the DD case it is negative for every temperature and every d (the McDonald functions, K_ν, are decreasing functions of their argument). Hence the Casimir force on the hyperplanes $x_N = 0, d$ is attractive while for the DN case the pressure on the boundary is positive for any $\beta \geq 0$ and $d > 0$, which implies repulsive force.

In the case of one spatial dimension ($N = 1$) there are no transverse modes and the wave vector $\vec{k}$ has only one quantized component. Another way to obtain the expressions for the $N = 1$ case is to promote N to a continuous variable in (16), (17), (19), (20), represent it as $1 + \varepsilon$ ($0 < \varepsilon \ll 1$) and take the limit $\varepsilon \to 0$. The explicit expressions (which coincide with those obtained by direct calculation) are:

$$E_c^{(DD)}(d,T) = -\frac{\pi d}{\beta^2}\sum_{n=1}^{\infty}\sinh^{-2}\left(\frac{2d\pi n}{\beta}\right) \tag{22}$$

$$E_c^{(DN)}(d,T) = -\frac{\pi d}{\beta^2}\sum_{n=1}^{\infty}(-1)^n\sinh^{-2}\left(\frac{2d\pi n}{\beta}\right) \tag{23}$$

$$F_c^{(DD)}(d,T) = -\sum_{n=1}^{\infty}\frac{1}{\beta n}\frac{1}{\exp\left(\frac{4\pi nd}{\beta}\right) - 1} \tag{24}$$

$$F_c^{(DN)}(d,T) = -\sum_{n=1}^{\infty}(-1)^n\frac{1}{\beta n}\frac{1}{\exp\left(\frac{4\pi nd}{\beta}\right) - 1} \tag{25}$$

CASIMIR ENTROPY AND THE HIGH TEMPERATURE LIMIT

Once given the Casimir energy and the Casimir free energy of a system, we define the Casimir entropy [3],[7],[11],[23],[24],[25] by the equation

$$S_c = \frac{E_c - F_c}{T}. \tag{26}$$

At the high temperature limit ($\beta \to 0$) the terms which contain McDonald functions tend to zero; thus we obtain that the entropies at the high temperature limit in DD and DN cases for $N > 1$ are:

$$S_c^{(DD)}(d, \infty) = \frac{\Gamma\left(\frac{N}{2}\right) L^{N-1}}{\pi^{\frac{N}{2}} 2^N d^{N-1}} \zeta(N) = -\beta F_c^{(DD)}(d, \infty) \tag{27}$$

$$S_c^{(DN)}(d, \infty) = -\frac{\Gamma\left(\frac{N}{2}\right) L^{N-1}}{\pi^{\frac{N}{2}} 2^N d^{N-1}} \left(1 - \frac{1}{2^{N-1}}\right) \zeta(N) = \\ -\beta F_c^{(DN)}(d, \infty) \tag{28}$$

(Eq.(27) coincides with the Casimir free energy in the high temperature limit given in [19]). We see that the Casimir entropy depends on the geometry (in our case the separation d) and on the type of boundary conditions involved. Its sign is not restricted to be non-negative since the Casimir entropy is the difference of the entropies of the constrained system and of the free system and thus might be negative, as happens for the DN b.c.. Recall that in the classical limit the Casimir energy tends to zero due to equipartition, and therefore in this case the Casimir force depends on the Casimir entropy only. Hence in the classical limit, the sign of the entropy determines the sign of the Casimir force. For the two cases of DD and DN b.c., we see that the entropies for the two cases differ in sign, thus giving rise to forces of opposite signs. Let us observe that $N = 1$ is the only dimension for which the Casimir entropy at the high temperature limit does not depend on d and thus it is an irrelevant constant and the Casimir force is zero. In the $N = 1$ case the boundary consists of two points, which do not "produce" enough geometry for the Casimir entropy (which gives rise to Casimir force). Another interesting observation which is deduced from Eqs.(19), (20) and (21) is that the Casimir pressure $P_c(d, T)$ ($P_c^{(DD)}(d, T)$ or $P_c^{(DN)}(d, T)$) has a fixed sign for fixed d and any T for $N > 1$.

GREEN'S FUNCTION AT ZERO TEMPERATURE

Now we turn to the problem of the validity of the mode summation technique. The correlation function of a massless scalar field at two points separated by the hyperplane $x_N = 0$, on which the scalar field is subjected to Neumann boundary conditions ($\frac{\partial \phi}{\partial x_N}\Big|_{x_N=0} = 0$) is not zero but is given by

$$\langle T\phi(x)\phi(y)\rangle = \Delta_1(x - y) + \Delta_1(x - \tilde{y}) \\ \text{for } x_N > 0, \ y_N < 0 \tag{29}$$

where $\tilde{y} = (y_0, y_1, ..., y_{N-1}, -y_N)$ and $\Delta_1(x - y) = \langle 0|\phi(x)\phi(y)|0\rangle$ is the homogeneous solution of $\Box_x \Delta_1(x - y) = 0$ (in three spatial dimensions $\Delta_1(x) = -\frac{1}{2\pi^2} P.P.\frac{1}{x^2}$ [5]). Since the field operators at points on opposite sides of the Neumann hyperplane are correlated, it is not possible to expand the field into independent eigenmodes living on opposite sides of the

Neumann hyperplane. Therefore we should check our assumption of expanding the field into eigenmodes separately in each one of the regions in the DN case. In contrast, the correlation function at points on opposite sides of the Dirichlet hyperplane ($\phi|_{x_N=0} = 0$) is zero. In this case an expansion of the field into independent eigenmodes living on opposite sides of the Dirichlet hyperplane is obviously justified. The expression for the Feynman propagator (which is the time ordered VEV of the two point function $\langle 0| T\phi(x)\phi(y)|0\rangle = i\Delta_F(x-y)$ in N spatial dimensions) is given by [4]

$$\Delta_F(x,y)_{N,m} =$$
$$\frac{-i\pi}{(4\pi i)^{\frac{N+1}{2}}} \left(\frac{2m^2}{-\frac{1}{2}(x-y)^2 + i\varepsilon}\right)^{\frac{N-1}{4}} H^{(2)}_{\frac{N-1}{2}}[(m^2(x-y)^2 - i\varepsilon)^{\frac{1}{2}}] \tag{30}$$

where $(x-y)^2 = \eta_{\alpha\beta}(x^\alpha - y^\alpha)(x^\beta - y^\beta)$ with $\eta_{\alpha\beta}$ the diagonal metric of the Minkowski flat space-time with the following components $\eta_{00} = -\eta_{11} = ... = -\eta_{NN} = 1$. $H^{(2)}_\nu(x)$ is the Hankel function [2]. The leading terms of $\Delta_F(x,y)_{N,m}$ as $m \to 0$ is found to be

$$\Delta_F(x,y)_{N>1,m\to 0} = \frac{2^{N-1}\Gamma\left(\frac{N-1}{2}\right)}{i^N (4\pi)^{\frac{N+1}{2}}} \left(\frac{1}{(x-y)^2}\right)^{\frac{N-1}{2}} \tag{31}$$

$$\Delta_F(x,y)_{N=1,m\to 0} = \frac{i}{4\pi} \ln[m^2(x-y)^2] \tag{32}$$

For $N > 1$ it is m independent and by substituting $N = 3$ into (31) we obtain the well known result [5]. For $N = 1$ we see that as $m \to 0$ we obtain an infra-red logarithmic divergent term (infra-red divergence). Actually we may obtain Eq.(32) if we promote N to a continuous variable ($N = 1 + \varepsilon$ ($\varepsilon \ll 1$)) in Eq.(31) and keep the leading terms for small ε (up to d-independent divergent terms)

$$\Delta_F(x,y)_{N>1,\varepsilon\to 0} = \frac{\Gamma\left(\frac{\varepsilon}{2}\right)}{4\pi i} \left(\frac{1}{(x-y)^2}\right)^{\frac{\varepsilon}{2}} =$$
$$\frac{\Gamma\left(\frac{\varepsilon}{2}\right)}{4\pi i} \exp(-\frac{\varepsilon}{2}\log(x-y)^2) = \frac{i}{4\pi}\log(\varepsilon^2(x-y)^2) + O(\varepsilon) \tag{33}$$

We see that Eq.(33) coincides with Eq.(32) if we take $\varepsilon = m$.

The free Feynman propagator in $N > 1$ spatial dimensions is given by Eq.(31). Green's functions for DD, DN, NN b.c. in the region between the planes $0 \le x_N, y_N \le d$ are given by Eq.(34), (35), Eq.(36) and in the section "Optical Green's function" (below) we show how we construct them. (Eq.(34) was previously obtained in [8].)

$$G^{(DD)}(x,y) =$$
$$\sum_{n=-\infty}^{\infty} (\Delta_F(x-y+2nd\hat{z}) - \Delta_F(x-\tilde{y}+2nd\hat{z})) \tag{34}$$

$$G^{(DN)}(x,y) =$$
$$\sum_{n=-\infty}^{\infty} (-1)^n (\Delta_F(x-y+2nd\hat{z}) - \Delta_F(x-\tilde{y}+2nd\hat{z})) \tag{35}$$

$$G^{(NN)}(x,y) =$$
$$\sum_{n=-\infty}^{\infty} (\Delta_F(x-y+2nd\hat{z}) + \Delta_F(x-\tilde{y}+2nd\hat{z})) \tag{36}$$

Once the Green's function with the proper b.c. is obtained, we may use it to express the VEV of the energy density of the scalar field in the following way

$$\langle h \rangle = \tfrac{1}{2} \left\langle \dot{\phi}^2 (x) + \left(\vec{\nabla} \phi (x) \right)^2 \right\rangle =$$
$$\tfrac{1}{2} \lim_{\varepsilon \to 0} [\partial_{x_0} \partial_{y_0} + ... + \partial_{x_N} \partial_{y_N}] \langle T \phi (x) \phi (y) \rangle \tag{37}$$

where in the last line we regularized the hamiltonian density operator using the point splitting $x^\mu = y^\mu + \varepsilon^\mu$. The VEV of the total energy in volume V is:

$$\langle H \rangle_V = \int_V \langle h \rangle \, dV \tag{38}$$

The Casimir energy density is, by definition

$$\langle h \rangle_c = \langle h \rangle - \langle h \rangle_{free} \tag{39}$$

where $\langle h \rangle$ is given by Eq.(37) for the constrained field and $\langle h \rangle_{free}$ is the energy density of the free field. The subtraction in (39) removes the short distance local divergences in (37), which cannot depend on the boundary conditions. The Casimir energy density is the VEV of the energy density of the field which is constrained by some boundary conditions, measured relative to the energy density of a free field. The Casimir energy is given by

$$E_c = \left(\int_V \langle h \rangle_c (x) \, dV \right)_{reg} + (d \text{ independent}), \tag{40}$$

where V is the region between the hyperplanes $(0 < x_N < d)$, and the (maybe infinite) d-independent contribution comes from the infinite region outside the hyperplanes. Henceforth we shall drop the d-independent contribution, which does not affect the Casimir force. In our three cases the Green's functions in the region $0 \leq x_N, y_N \leq d$ are infinite series (sum over index n) which contain the terms $\Delta_F (x - y + 2nd\hat{z})$ and $\Delta_F (x - \tilde{y} + 2nd\hat{z})$ with different coefficients, appropriate to the given case. Substitute the expression for the Green's function into Eq.(37). If we interchange summation over n and integration with differentiation with respect to the coordinates we find the contribution of each term to the Casimir energy. We use the coordinate representation of the free propagator Δ_F given by (31), (32). The $n = 0$ term which corresponds to the free propagator is cancelled by Eq.(39). Each "even path" (in the next section "Optical Green's function", we will explain the terminology "even" and "odd" paths) which is represented by the term $\Delta_F (x - y + 2nd\hat{z})$ $(n \neq 0)$ contributes to the Casimir energy

$$-\frac{\Gamma \left(\frac{N+1}{2} \right)}{2\pi^{\frac{N+1}{2}}} \frac{1}{l_n^{N+1}} \tag{41}$$

while each "odd path" which is represented by the term $\Delta_F (x - \tilde{y} + 2nd\hat{z})$ contributes

$$-\frac{\Gamma \left(\frac{N+1}{2} \right)}{2\pi^{\frac{N+1}{2}}} \frac{1}{l_n^{N+1}} \tag{42}$$

where l_n is the length of the path labelled by n (n may be any integer) and is given by

$$l_n = \begin{cases} |2dn| & \text{for "even path"} \\ |2x_N + 2dn| & \text{for "odd path"} \end{cases} \tag{43}$$

10

The overall contribution of the "odd" paths to the Casimir energy is d-independent as one may verify for all three cases of boundary conditions and therefore unimportant. Consequently, summing over the index n we obtain the following well known expressions for Casimir energy (for any N) [19], [26] which coincide with Eq.(11) and Eq.(12).

$$E_c^{(DD)}(d,0) = -\frac{\Gamma\left(\frac{N+1}{2}\right) L^{N-1}}{(4\pi)^{\frac{N+1}{2}} d^N} \zeta(N+1) \tag{44}$$

$$E_c^{(DN)}(d,0) = -\left(1 - \frac{1}{2^N}\right) E_c^{(DD)}(d,0) \tag{45}$$

$$E_c^{(NN)}(d,0) = E_c^{(DD)}(d,0) \tag{46}$$

OPTICAL GREEN'S FUNCTION

In this section we are mostly inspired by [15],[16] (which construct Green's functions for various geometries via classical trajectories of the rays of light) and [17]. Let us consider a massless scalar field which is subjected to Dirichlet b.c. on the first hyperplane at $x_N = 0$ and Neumann b.c. on the second hyperplane at $x_N = d$. Our goal in this section is to show how one may write the exact Green's function in the region between the two hyperplanes by making classical, geometric optical considerations. We shall express the Green's function between the hyperplanes in terms of propagators connecting two points via paths which hit the hyperplanes and are reflected (according to the laws of geometric optics). For each time that the path hits the Dirichlet hyperplane we multiply the propagator by (-1) while each time it hits the Neumann hyperplane we multiply it by 1. Formally we characterize each path by an index $r = \left(\sigma_D^{(r)}, \sigma_N^{(r)}\right)$ where $\sigma_D^{(r)}$ and $\sigma_N^{(r)}$ are the reflection numbers, the number of times a given path r is reflected from the first and second hyperplane, respectively. The total number of reflections $\sigma_D^{(r)} + \sigma_N^{(r)}$ for path r is denoted by $|r|$. Obviously for every classical path which we consider, the following relation holds

$$\left|\sigma_D^{(r)} - \sigma_N^{(r)}\right| \leq 1. \tag{47}$$

It is convenient to separate the paths into "even paths" and "odd paths". "Even paths" means paths which are reflected an even number of times from the hyperplanes. For these paths the sum of reflection numbers, $|r|$, is even and $\sigma_D^{(r)} = \sigma_N^{(r)}$ (The last equality is consistent with (47)). "Odd paths" means paths reflected an odd number of times from the hyperplanes. For odd paths, by definition, $|r|$ is odd and we may divide them into two classes according to $\sigma_D^{(r)} - \sigma_N^{(r)} = \pm 1$. The total energy is obtained by summation over all the contributions of all paths. The total energy due to "even paths" for which $|r| = 2n$ is:

$$E_{even}^{(DN)} = i \int_V dV \Delta_F(x-y) +$$

$$2i \sum_{n=1}^{\infty} (-1)^n \int_V dV \Delta_F(x-y+2nd\hat{z}) = i \sum_{n=-\infty}^{\infty} (-1)^n \int_V dV \Delta_F(x-y+2nd\hat{z}) \tag{48}$$

11

The first factor in Eq.(48) corresponds to the path $|r| = 0$. The factor 2 is due to degeneracy of the even paths for which $|r| \neq 0$. Together they combine to the r.h.s of Eq.(48). The odd paths which first hit the first hyperplane (the class $\sigma_D^{(r)} - \sigma_N^{(r)} = 1$) contribute

$$i \sum_{n=0}^{\infty} (-1)^{n+1} \int_V dV \Delta_F \left(x - \tilde{y} + 2nd\hat{z} \right) \tag{49}$$

while those which hit the second hyperplane first (the class $\sigma_D^{(r)} - \sigma_N^{(r)} = -1$) contribute

$$i \sum_{n=1}^{\infty} (-1)^{n+1} \int_V dV \Delta_F \left(x - \tilde{y} - 2nd\hat{z} \right) \tag{50}$$

Together they give rise to

$$E_{odd}^{(DN)} = i \sum_{n=-\infty}^{\infty} (-1)^{n+1} \int_V dV \Delta_F \left(x - \tilde{y} + 2nd\hat{z} \right) \tag{51}$$

The contribution to the Casimir energy of the odd paths is actually zero ($E_{odd}^{(DN)} = 0$), as we already checked in the previous section, and only even paths contribute to it. It is given by

$$E_c^{(DN)} = i \sum_{n=-\infty}^{\infty}{}' (-1)^n \int_V dV \Delta_F \left(x - y + 2nd\hat{z} \right) \tag{52}$$

(where the prime means taking the sum without the $n = 0$ term).

One can also easily obtain $E_c^{(DD)}$, $E_c^{(NN)}$ which coincide with the known results.

SUMMARY

To summarize, we presented in this work calculations for the Casimir energy, free energy and entropy of a scalar massless field at finite temperature in the cases of DD and DN boundary conditions. We used the technique of mode summation for any temperature and also the Green's function method for zero temperature. In the case of DD boundary conditions we used the usual Poisson summation formula while in the case of DN boundary conditions we had to modify it and wrote it in a slightly different form. Later we used an identity for infinite series to express our results by means of modified Bessel functions of the second kind (McDonald functions). The new results are the Casimir energy, Casimir free energy and Casimir entropy due to DN b.c.. At the high temperature limit (namely when $T \gg T_c$) we obtain that Kirchhoff's law holds, namely, that the Casimir energy tends to zero in both cases, thus it is independent of the boundary conditions. Another result which we obtain is that at the high temperature limit the force between the hyperplanes is attractive for the DD boundary conditions and repulsive in the case of DN boundary conditions. The fact that the Casimir energy at high temperature limit is zero but the Casimir force does not vanish indicates that the Casimir force in the high temperature limit is purely entropic.

The results we obtained in the zero temperature by using Green's functions coincide with those performed by the mode summation. Further research is needed to understand why does the mode summation, which inherently assumes non-correlation between the modes, work in the DN case.

ACKNOWLEDGMENTS

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